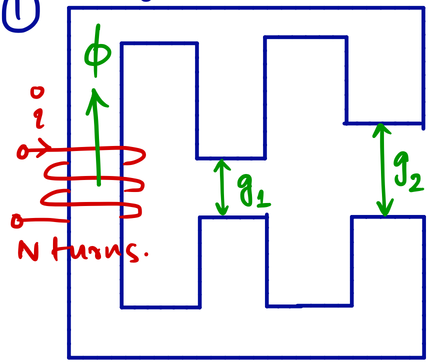
# Magnetic circuit examples

(1)




Parameters of the arrangement:

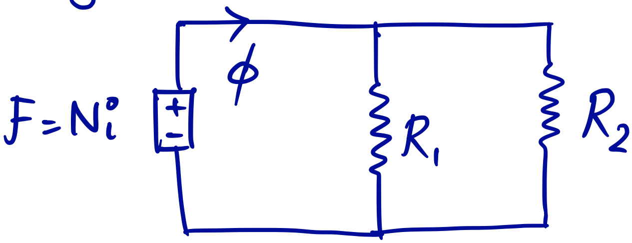
- $\mu_{core} = \infty$
- Uniform cross-sectional area of $1\ cm^2$.
- $g_1 = 1$ mm, $g_2 = 2$ mm
- $N = 200$ turns.
- No fringing in air gaps.

Compute the self-inductance.

Steps: 1. Draw magnetic circuit and calculate the magnetic flux $\phi$.

2. Compute self-inductance $\lambda$, that is proportional to $i$. The proportionality constant is self-inductance $\mathcal{L}$.

Magnetic circuit :




$$\mathcal{R}_1 = \frac{g_1}{\mu_0 \cdot A_{core}} = \frac{1\times10^{-3}}{4\pi\times10^{-7}\times1\times10^{-4}}\ \frac{A\text{-}t}{Wb}.$$

$$= 8\times10^{6}\ At/Wb.$$

$$\mathcal{R}_2 = \frac{g_2}{\mu_0 A_{core}} = 2\mathcal{R}_1 = 1.6\times10^{7}\ A\text{-}t/Wb.$$

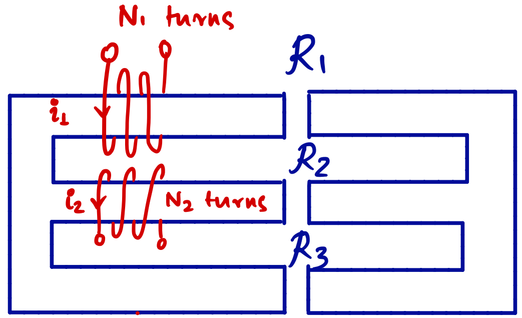
$$\boxed{\begin{aligned}\mathcal{R}_1 \| \mathcal{R}_2 &= \mathcal{R}_1 \| 2\mathcal{R}_1\\ &= \frac{2\mathcal{R}_1^2}{3\mathcal{R}_1}\\ &= \frac{2}{3}\mathcal{R}_1.\end{aligned}}$$

$$\therefore\ \phi = \frac{Ni}{\mathcal{R}_1 \| \mathcal{R}_2}$$

$$\Rightarrow \lambda = N\phi = \underbrace{\left(\frac{N^2}{\frac{2}{3}\mathcal{R}_1}\right)}_{\therefore = L}\cdot i$$

$$\therefore\ L = \frac{200\times200}{\frac{2}{3}\times8\times10^{6}}\ H = 7.5\ mH.$$

(2)



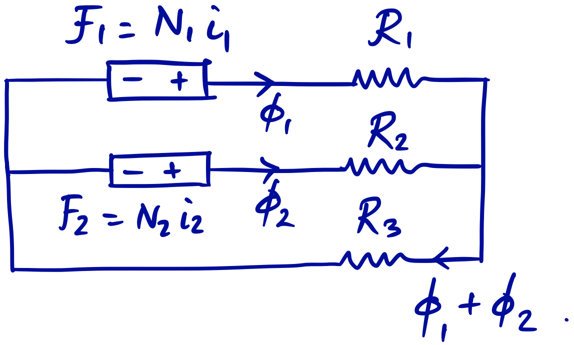

Assume $\mu_{core} = \infty$.

In the arrangement shown above, let the reluctances of the air gaps be shown as above. Compute the mutual inductance of the coils.

New term!

Mutual inductance: How does the flux linkage of the first coil vary with the current in the other coil?

Let's compute it using a magnetic circuit.




- Let's calculate $\phi_1$ and $\phi_2$ in terms of $i_1, i_2$, $R_1, R_2, R_3$.

$$N_1 i_1 = \phi_1 R_1 + (\phi_1 + \phi_2) R_3 ,$$

$$N_2 i_2 = \phi_2 R_2 + (\phi_1 + \phi_2) R_3 .$$

An useful trick to solve simultaneous equations in 2 variables.

$$\Rightarrow \underbrace{\begin{pmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{pmatrix}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}^{-1} \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix} .$$

<u>Digression:</u> $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

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$$\therefore \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} R_1+R_3 & R_3 \\ R_3 & R_2+R_3 \end{pmatrix}^{-1} \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

$$= \frac{1}{(R_1+R_3)(R_2+R_3)-R_3^2}\begin{pmatrix} R_2+R_3 & -R_3 \\ -R_3 & R_1+R_3 \end{pmatrix}\begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

$$= \frac{1}{R_1R_2+R_3(R_1+R_2)}\begin{pmatrix} N_1(R_2+R_3)\,i_1 - N_2 R_3\, i_2 \\ -N_1 R_3\, i_1 + N_2(R_1+R_3)\,i_2 \end{pmatrix}$$

$$\therefore \lambda_1 = N_1\phi_1$$

$$= \frac{N_1^2(R_2+R_3)}{R_1R_2+R_3(R_1+R_2)}\,i_1 - \frac{N_1N_2R_3}{R_1R_2+R_3(R_1+R_2)}\,i_2,$$

$$\lambda_2 = N_2\phi_2$$

$$= -\frac{N_1N_2R_3}{R_1R_2+R_3(R_1+R_2)}\,i_1 + \frac{N_2^2(R_1+R_3)}{R_1R_2+R_3(R_1+R_2)}\,i_2.$$

$$\boxed{\begin{aligned}\lambda_1 &= \underbrace{\left(\frac{N_1^2(R_2+R_3)}{R_1R_2+R_3(R_1+R_2)}\right)}_{:=\mathcal{L}_{11}} i_1 + \underbrace{\left(-\frac{N_1N_2R_3}{R_1R_2+R_3(R_1+R_2)}\right)}_{:=\mathcal{L}_{12}} i_2, \\ \lambda_2 &= \underbrace{\left(-\frac{N_1N_2R_3}{R_1R_2+R_3(R_1+R_2)}\right)}_{:=\mathcal{L}_{21}} i_1 + \underbrace{\left(\frac{N_2^2(R_1+R_3)}{R_1R_2+R_3(R_1+R_2)}\right)}_{:=\mathcal{L}_{22}} i_2.\end{aligned}}$$

$\mathcal{L}_{11}$ = self inductance of coil 1.

$\mathcal{L}_{22}$ = self inductance of coil 2.

$\mathcal{L}_{12}$, $\mathcal{L}_{21}$ = mutual inductance.

With an abuse of notation, we usually call $\mathcal{M} = |\mathcal{L}_{12}| = |\mathcal{L}_{21}|$ as the mutual inductance. How $\lambda$ depends on the mutual inductance is decided through polarity markings, described below.

- $k = \dfrac{M}{\sqrt{L_1 L_2}}$ is called the coupling coeff.

When $k$ is high (close to 1), we call these coils tightly coupled. When $k$ is closer to zero, they are loosely coupled.

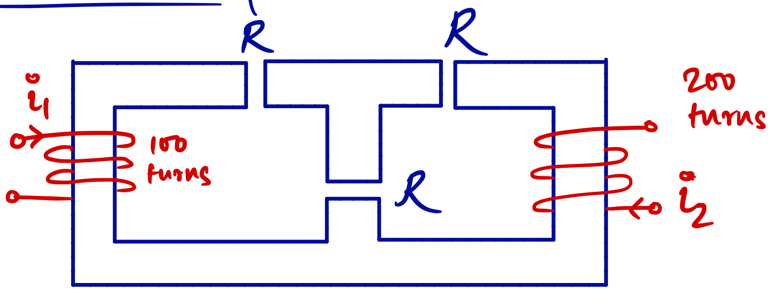
- We obtained expressions of the form

$$\lambda_1 = \mathcal{L}_{11} i_1 \pm \mathcal{M} i_2,$$
$$\lambda_2 = \pm \mathcal{M} i_1 + \mathcal{L}_{22} i_2.$$

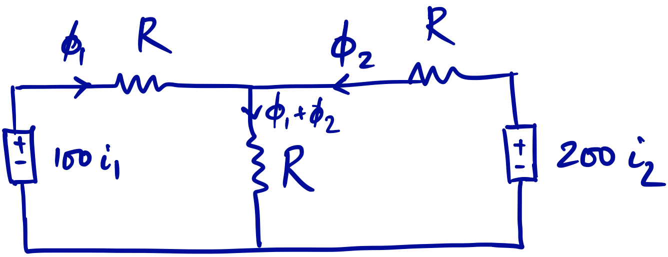
$$\Rightarrow \quad v_1 = \mathcal{L}_{11} \frac{di_1}{dt} + \mathcal{M} \frac{di_2}{dt},$$

$$v_2 = \pm \mathcal{M} \frac{di_1}{dt} + \mathcal{L}_{22} \frac{di_2}{dt}.$$

Another example.




Suppose $\mu_{core} = \infty$, and $\mathcal{R} = 10^6$ A.t/Wb.
Compute the self and mutual inductances of the two coils. Compute the coupling coefficient.




$$100\, i_1 = R\phi_1 + R(\phi_1+\phi_2),$$

$$200\, i_2 = R\phi_2 + R(\phi_1+\phi_2).$$

$$\Rightarrow \begin{pmatrix} 2R & R \\ R & 2R \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 100\, i_1 \\ 200\, i_2 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 2R & R \\ R & 2R \end{pmatrix}^{-1} \begin{pmatrix} 100\, i_1 \\ 200\, i_2 \end{pmatrix}$$

$$= \frac{1}{(2R)(2R) - R^2} \cdot \begin{pmatrix} 2R & -R \\ -R & 2R \end{pmatrix} \begin{pmatrix} 100\, i_1 \\ 200\, i_2 \end{pmatrix}$$

$$= \frac{1}{3R^2} \begin{pmatrix} 200 R i_1 - 200 R i_2 \\ -100 R i_1 + 400 R i_2 \end{pmatrix} .$$

$$= \begin{pmatrix} \frac{200}{3R} \cdot i_1 - \frac{200}{3R} i_2 \\ -\frac{100}{3R} i_1 + \frac{400}{3R} i_2 \end{pmatrix} .$$

$$\Rightarrow \lambda_1 = N_1 \cdot \phi_1 = \frac{100 \times 200}{3R} i_1 - \frac{100 \times 200}{3R} i_2 ,$$

and $\lambda_2 = N_2 \cdot \phi_2 = \frac{-100 \times 200}{3R} i_1 + \frac{400 \times 200}{3R} i_2 .$

Upon substituting $R = 10^6$ At/Wb, we get

$$\mathcal{L}_{11} = 6.6 \text{ mH}, \quad \mathcal{L}_{22} = 26.6 \text{ mH},$$

$$\mathcal{M} = 6.6 \text{ mH}.$$

$$k = \frac{\mathcal{M}}{\sqrt{\mathcal{L}_{11}\mathcal{L}_{22}}} = \frac{6.6}{\sqrt{6.6 \times 266}} = \sqrt{\frac{6.6}{26.6}} = 0.498$$

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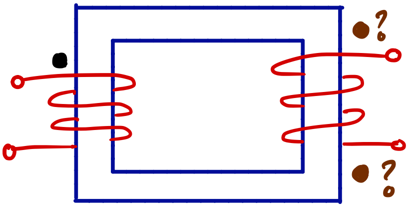
## Polarity markings

These are dots put on the coils to indicate whether $+\mathcal{M}$ or $-\mathcal{M}$ features in the voltage current relationship.

**Agenda:**

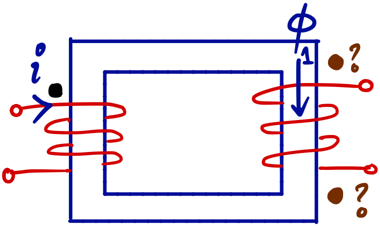
1. Given an arrangement, how to draw polarity markings.
2. Given polarity markings, how to write "loop equations!"

① Drawing polarity markings. Consider an example. Put the first dot on either end of the first coil as shown. Let's study where to put the dot on the second coil.

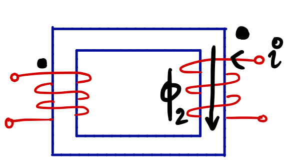


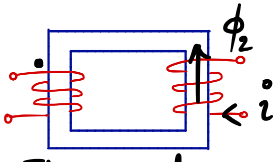

**Algorithm:**

Imagine a current going into coil at the dotted end as shown. Determine the direction of flux due to that current in the other coil as shown.




Ask the question: Which end of the second coil should we drive current into that induces flux in that coil in the same direction?



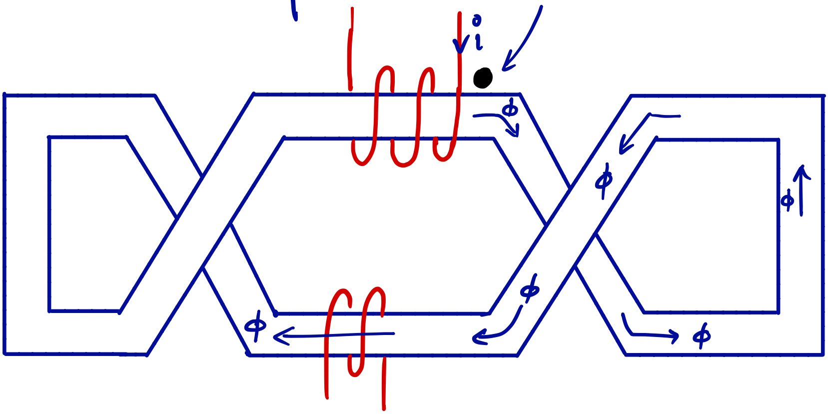



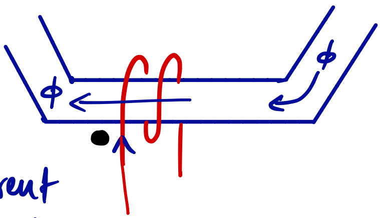

This one does not match.

Put dot where the fluxes $\phi_1$ and $\phi_2$ are in the same direction.

## Another example:

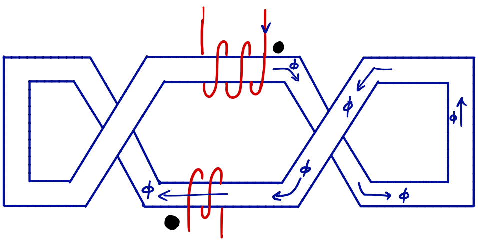
Put this arbitrarily.




Driving current this way produces flux in the same direction.

$\Rightarrow$ Put dot as shown.




Resulting polarity markings.

② Deriving circuit equations or loop equations, given polarity markings.

Follow the rule below :-

(Voltage difference between dotted end & non-dotted end of a coil)

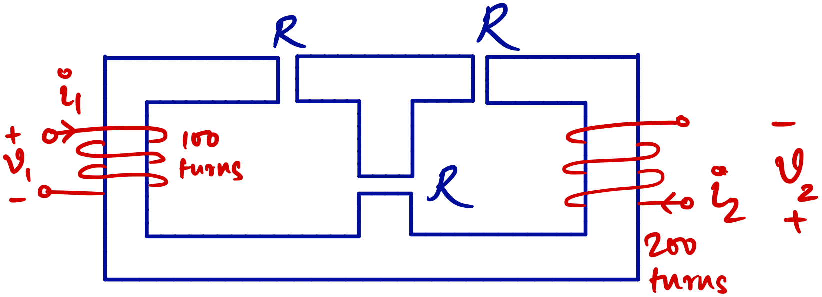
$= \mathcal{L} \dfrac{d}{dt}$ (current entering the dotted end of the same coil)

$+ \mathcal{M} \dfrac{d}{dt}$ (current entering the dotted end of the other coil).

Here, $\mathcal{L}$ = self-inductance of the coil in question.

$\mathcal{M}$ = mutual inductance between the coils.

## Examples :-




$\mathcal{L}_{11} = 6.6$ mH,

$\mathcal{L}_{22} = 26.6$ mH,

$\mathcal{M} = 6.6$ mH.

Consider the above arrangement that we have seen before, for which we computed the inductances.

① Put polarity markings.

② Derive $v_1$ and $v_2$ in terms of $i_1, i_2, \mathcal{L}_{11}, \mathcal{L}_{22}, \mathcal{M}$.

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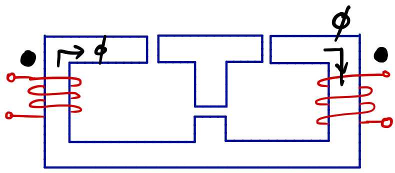
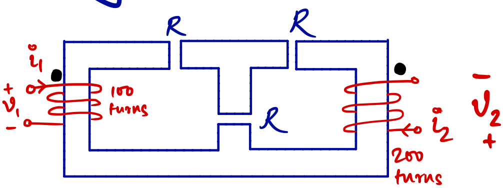
Answering part ①: Erase all currents/voltages. Polarity markings only depend on the geometry.




Verify the figure to your left.

Answering part ②: Start with the diagram with your polarity markings. Follow the rule to relate voltages & currents.




$$v_1 = \mathcal{L}_{11} \cdot \frac{d}{dt}(i_1) + \mathcal{M}\frac{d}{dt}(-i_2)$$

$$(-v_2) = \mathcal{L}_{22}\frac{d}{dt}(-i_2) + \mathcal{M}\frac{d}{dt}(i_1).$$

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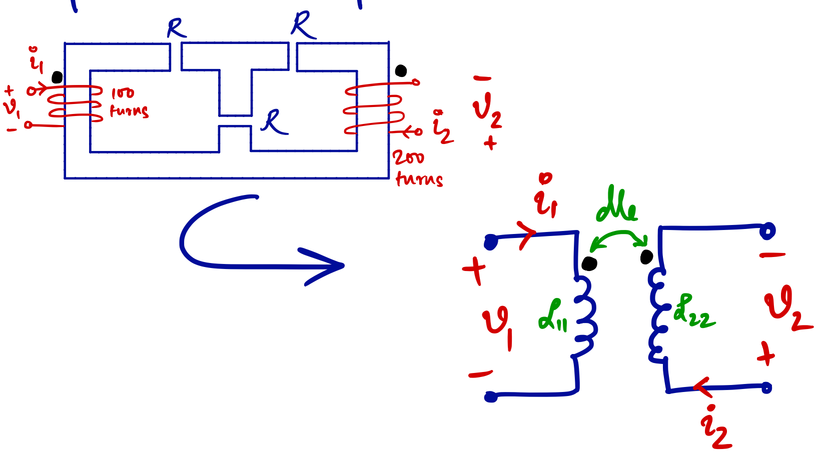
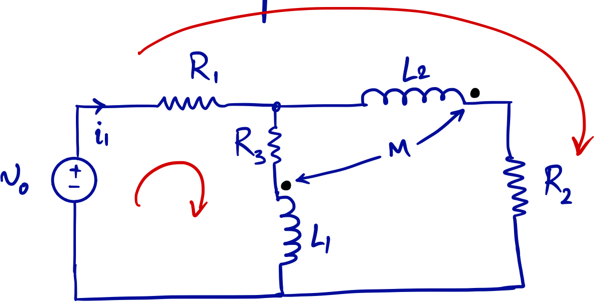
Coupled coil representation:

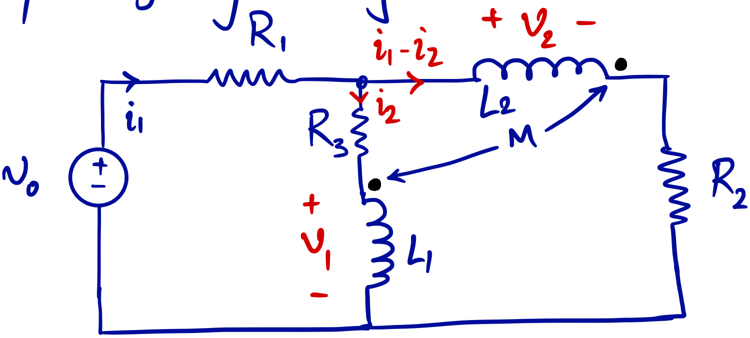


An example with coupled coils:




Write down KVL around the two paths shown in red.

Start by assigning voltages and currents.




- $v_0 - i_1 R_1 - i_2 R_3 - v_1 = 0$
- $v_0 - i_1 R_1 - v_2 - (i_1 - i_2) R_2 = 0.$
- $v_1 = L_1 \frac{d}{dt}(i_2) + M \frac{d}{dt}(i_2 - i_1)$

notice the sign.

- $-v_2 = L_2 \frac{d}{dt}(i_2 - i_1) + M \frac{d}{dt} i_2.$

- Now, suppose $v_o$ is sinusoidal, whose phasor is given by $\bar{V}_o$. Write the same KVL's with phasors.

$$\boxed{\begin{aligned} &\bullet\ v_o - i_1R_1 - i_2R_3 - v_1 = 0 \\ &\bullet\ v_o - i_1R_1 - v_2 - (i_1 - i_2)R_2 = 0. \\ &\bullet\ v_1 = L_1 \frac{d}{dt}(i_2) + M\frac{d}{dt}(i_2 - i_1) \\ &\bullet\ -v_2 = L_2\frac{d}{dt}(i_2 - i_1) + M\frac{d}{dt} i_2. \end{aligned}}$$

This is what we derived before. Let's write the phasor eq.

- $\bar{V}_o - \bar{I}_1 R_1 - \bar{I}_2 R_3 - \bar{V}_1 = 0$
- $\bar{V}_o - \bar{I}_1 R_1 - \bar{V}_2 - (\bar{I}_1 - \bar{I}_2) R_2 = 0$
- $\bar{V}_1 = j\omega L_1 \bar{I}_2 + j\omega M (\bar{I}_2 - \bar{I}_1)$
- $-\bar{V}_2 = j\omega L_2 (\bar{I}_2 - \bar{I}_1) + j\omega M \bar{I}_2.$

<u>Exercise</u>: You can solve for all currents $\bar{I}_1, \bar{I}_2$ and voltages $\bar{V}_1, \bar{V}_2$ in terms of $\bar{V}_o$. Do it!

- So far, we have studied how to derive a coupled coil representation of an arrangement (a transformer) from its geometry and write KVL with it.

- Next on the agenda: Draw an inductor + ideal transformer representation of a transformer.

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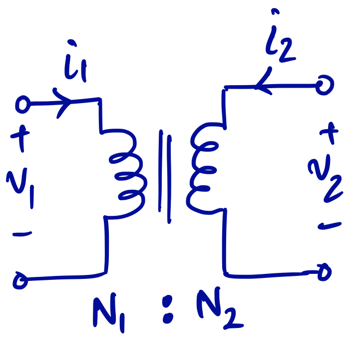
## Ideal transformer:




An ideal transformer is a circuit component whose i-v characteristics are given by

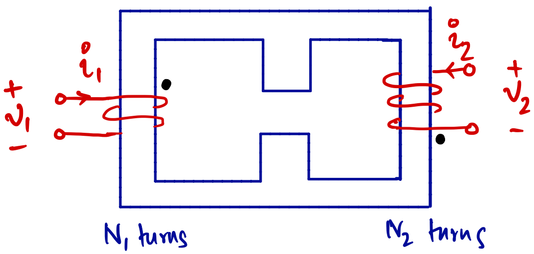
$$\frac{v_1}{v_2} = \frac{N_1}{N_2}; \quad \frac{i_1}{i_2} = -\frac{N_2}{N_1}.$$

• Can we conceptualize an ideal transformer from two coils wound around a core?

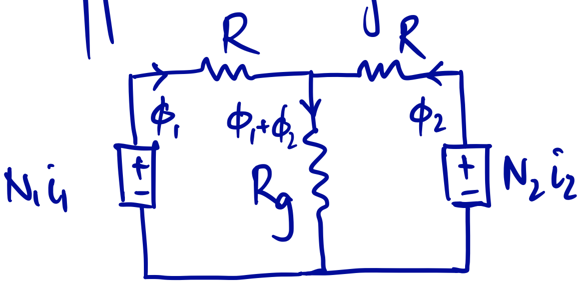
Yes! You can. Let's derive it

This part is optional. Skip 5 pages including this one.

Consider the arrangement shown.




Suppose its magnetic circuit is given by




Let's compute $\phi_1$, $\phi_2$, $\phi_1+\phi_2$, and $v_1, v_2$.

$$N_1 i_1 = R\phi_1 + R_g(\phi_1 + \phi_2).$$

$$N_2 i_2 = R\phi_2 + R_g(\phi_1 + \phi_2).$$

$$\Rightarrow \begin{pmatrix} R+R_g & R_g \\ R_g & R+R_g \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} R+R_g & R_g \\ R_g & R+R_g \end{pmatrix}^{-1} \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

$$= \frac{1}{(R+R_g)^2 - R_g^2} \begin{pmatrix} R+R_g & -R_g \\ -R_g & R+R_g \end{pmatrix} \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

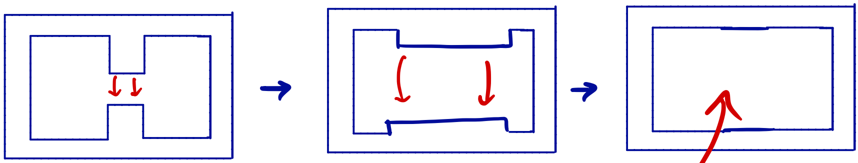
$$= \frac{1}{(R+R_g)^2 - R_g^2} \begin{pmatrix} (R+R_g) N_1 i_1 - R_g N_2 i_2 \\ -R_g N_1 i_1 + (R+R_g) N_2 i_2 \end{pmatrix}$$

$$\Rightarrow \phi_1 + \phi_2 = \frac{1}{(R+R_g)^2 - R_g^2} \left[ R N_1 i_1 + R N_2 i_2 \right].$$

$$\Rightarrow \quad \phi_1 + \phi_2 = \frac{R}{(R+R_g)^2 - R_g^2} \left( N_1 i_1 + N_2 i_2 \right).$$

Let's analyze the result as $R \downarrow 0$, $R_g \uparrow \infty$.

- $R_g$ really large can be thought of as making the middle column wider & smaller..



Almost no "leakage" flux flowing outside of core.

- $R$ very small $\equiv$ $\mu_{core}$ really large.

Then, $$\phi_1 + \phi_2 = \frac{R}{R^2 + 2RR_g} \cdot \left( N_1 i_1 + N_2 i_2 \right)$$

$$= \frac{1}{R + 2R_g} \left( N_1 i_1 + N_2 i_2 \right)$$

$$\approx 0.$$

$$\phi_1 + \phi_2 \approx 0 \Rightarrow N_1 i_1 + N_2 i_2 \approx 0.$$

Also, $$\phi_1 = \frac{1}{(R+R_g)^2 - R_g^2}\left[(R+R_g) N_1 i_1 - R_g N_2 i_2\right]$$

$$= \frac{R+R_g}{R^2+2RR_g} N_1 i_1 - \frac{R_g}{R^2+2RR_g} N_2 i_2 .$$

$$= \frac{1}{R}\left[\underbrace{\frac{R+R_g}{R+2R_g}}_{\approx \frac{1}{2}} N_1 i_1 - \underbrace{\frac{R_g}{R+2R_g}}_{\frac{1}{2}} N_2 i_2\right]$$

$$\approx \frac{1}{2R}\left(N_1 i_1 - N_2 i_2\right).$$

Similarly, $$\phi_2 = \frac{1}{2R}\left(N_2 i_1 - N_1 i_1\right) \approx -\phi_1 .$$

$$v_1 = N_1 \frac{d\phi_1}{dt}, \quad v_2 = -N_2 \frac{d\phi_2}{dt}.$$ (why -ve? check polarity marking)

$$\Rightarrow \frac{v_1}{v_2} = \frac{N_1}{N_2} \cdot \frac{d\phi_1/dt}{-d\phi_2/dt} \approx N_1/N_2 .$$

- What is self and mutual inductances of an ideal transformer?

$$\mathcal{L}_1 = \frac{R+R_g}{R^2+2RR_g} N_1^2$$ from the above derivation.

$$= \underbrace{\frac{R+R_g}{R+2R_g}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{N_1^2}{R}}_{\to \infty} \longrightarrow \infty.$$

$\to \infty$. as $R_g \uparrow \infty$, $R \downarrow 0$.

Similarly $\mathcal{L}_2 \to \infty$.

$$M = \frac{R_g N_1 N_2}{R^2+2RR_g} = \frac{R_g}{R+R_g} \cdot \frac{N_1 N_2}{R} \longrightarrow \infty.$$

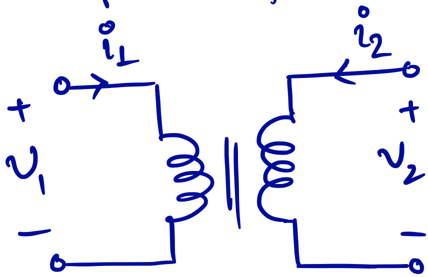
- Coupling coeff.

$$k = \frac{M}{\sqrt{\mathcal{L}_1 \mathcal{L}_2}} = \frac{\frac{R_g}{\cancel{R+R_g}} \cdot \frac{\cancel{N_1 N_2}}{\cancel{R}}}{\sqrt{\frac{R+R_g}{\cancel{R+2R_g}} \cdot \frac{\cancel{N_1^2}}{\cancel{R}} \cdot \frac{R+R_g}{\cancel{R+2R_g}} \cdot \frac{\cancel{N_2^2}}{\cancel{R}}}} = \frac{R_g}{R+R_g} \to 1.$$

$\therefore$ For an ideal transformer, $\mathcal{L}_1 = \mathcal{L}_2 = M = \infty$.

$k = 1$.

## Properties of ideal transformer:




$N_1 : N_2$

$\underbrace{\phantom{N_1 : N_2}}$ called "turns ratio", often denoted by 'a'.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a$$

$$\frac{i_1}{i_2} = -\frac{N_2}{N_1} = -\frac{1}{a}.$$

When direction of $i_2$ is reversed, then change sign on this relationship!.

Total power entereing the ideal transformer

$$= v_1(t) \cdot i_1(t) + v_2(t)\, i_2(t)$$

$$= v_1(t) \cdot i_1(t) + \left(\frac{v_1(t)}{\cancel{a}}\right)\left(-\cancel{a}\, i_1(t)\right)$$

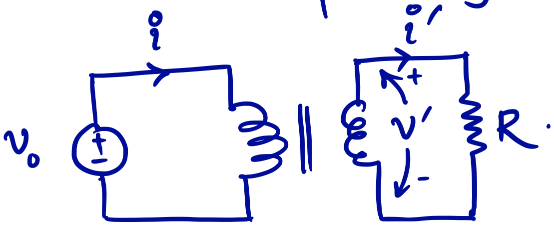
$$= v_1(t)\, i_1(t) - v_1(t)\, i_1(t)$$

$$= 0.$$

Ideal transformer is lossless.

## "Referring" a resistance.

Consider the following circuit:




Turns ratio $= a$.

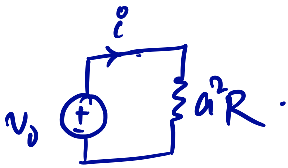
**Compute $i$.**

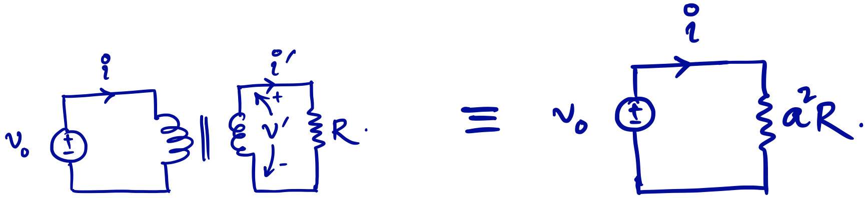
- $\frac{v_0}{v'} = a$, $\frac{i}{i'} = \frac{1}{a}$. (Notice ... no negative sign. ... Why? Check the direction of current.)
- $v' = i'R$.

$\therefore$ $v' = a \cdot iR \Rightarrow v_0 = av' = a \cdot (aiR) = i(a^2R)$

Notice $v_0 = i \cdot (a^2R)$

$\Rightarrow$ The resistance on the "secondary" side of the ideal transformer appears like a resistor with resistance $a^2R$.






Turns ratio $= a$.

In other words, we have referred $R$ to the "primary" side of the transformer.

**Convention:**

Side with the source = primary side

Transformer

Side with the load = secondary side